CHAPTER 2

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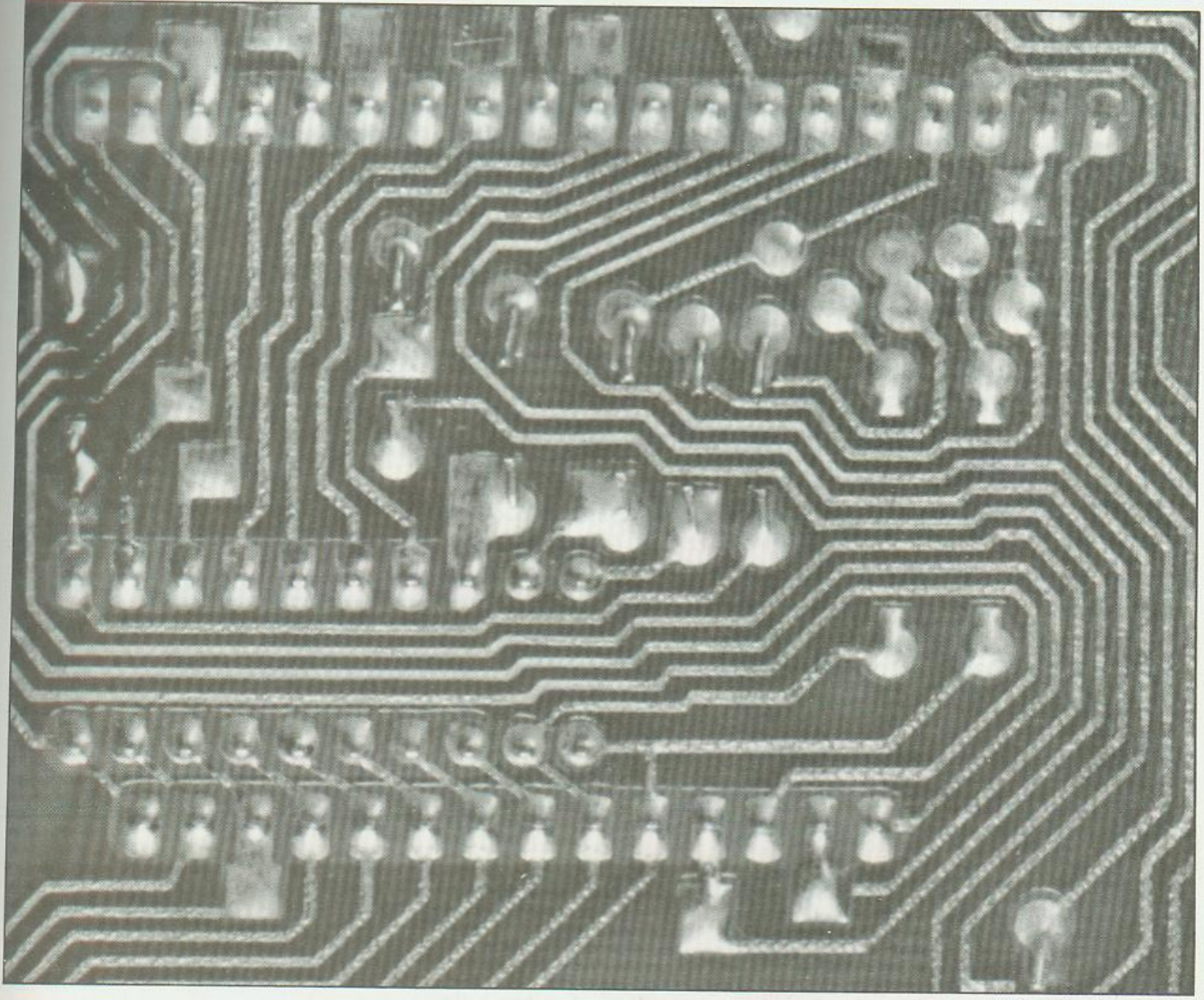
Reasoning in Geometry



2–1 The Process of Inductive Reasoning

THE SUN RISES. THE SUN SETS. THE SUN RISES. THE SUN SETS.
I CAN HARDLY WAIT TILL TOMORROW.
ANOTHER JAM-PACKED DAY OF CELESTIAL THRILLS.

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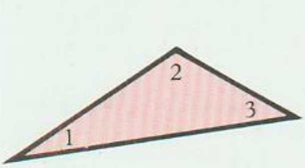
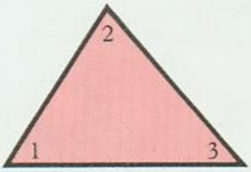
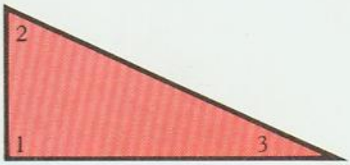
Reasoning is a process of drawing conclusions from information. Sometimes people draw conclusions based on their observations. After watching an event give the same results several times in succession, a person often concludes that the event will always have the same outcome. This kind of reasoning is called *inductive reasoning*. We shall call a conclusion arrived at through inductive reasoning a *generalization*.

The three examples that follow show how inductive reasoning can be used in geometry.

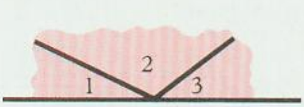
Example 1 Suppose someone has cut three different shaped triangles from a piece of paper.

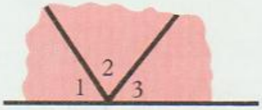


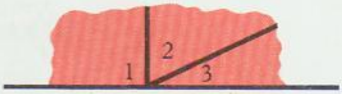




The corners of each of the triangles have been cut off and fitted together as shown below.





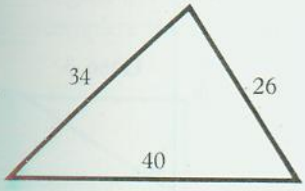


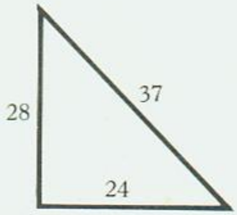
What do you observe about the sum of the angle measures? Do you think this will be true for all triangles?

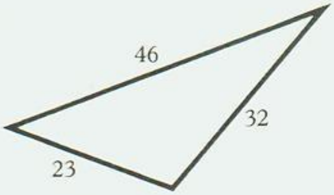
Complete this generalization:

The sum of the measures of the three angles of a triangle is $\underline{\ ?\ }$.

Example 2 Suppose someone has measured all three sides of three different triangles.





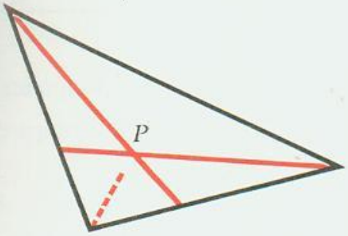


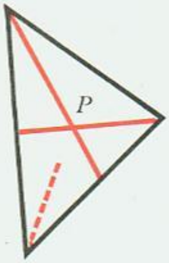
In these triangles the sum of the lengths of two sides is greater than the length of the third side. Do you think this will be true for all triangles?

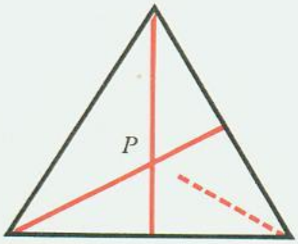
Complete this generalization:

The sum of the lengths of two sides of a triangle is _?_ the length of the third side.

Example 3 Suppose someone constructs the angle bisectors of each angle of three different triangles.







In each triangle will all three angle bisectors meet in point P?
Do you think this will be true for all triangles?

Complete this generalization:

The angle bisectors of a triangle _?_ at a point _?_ (inside, on, outside) the triangle.

The process of inductive reasoning may be described as stated here.

Inductive Reasoning

Step 1 You observe that a property is true for every case you check.

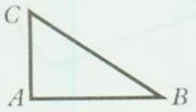
Step 2 Since the property is true for all cases checked, you conclude that the property is true for all other cases also and state a generalization.

EXERCISES

A.

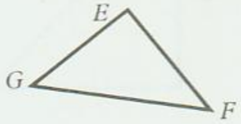
Complete the generalization in exercises 1 and 2.

1. **Case 1**



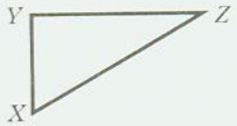
$\angle A$ is a right angle

Case 2



$\angle E$ is a right angle

Case 3

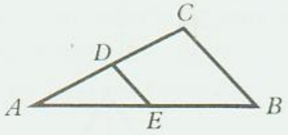


$\angle Y$ is a right angle

In each case, which side of the triangle is longest?

Generalization: In a right triangle, the side opposite the right angle is _?_ side.

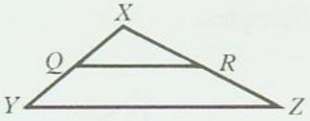
2. **Case 1**



D and E are midpoints

How does DE compare with CB?

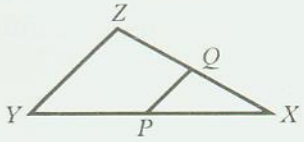
Case 2



Q and R are midpoints

How does QR compare with YZ?

Case 3



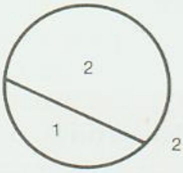
P and Q are midpoints

How does PQ compare with YZ?

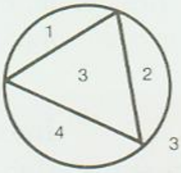
Generalization: The measure of a line segment joining the midpoints of two sides of a triangle is _?_ the third side.

Activity

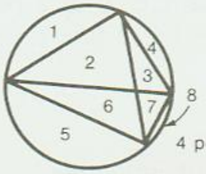
All chords determined by a set of points on a circle divide the interior of the circle into regions.



2 points



3 points



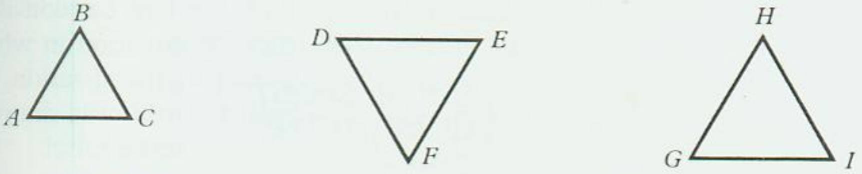
4 points

Number of Points	Maximum Number of Regions
1	0
2	2
3	4
4	8
5	—
6	—
7	—

1. Study the pattern in the table and predict the number of regions for 5, 6, and 7 points.
2. Check your prediction by drawing large figures (circles at least 20 cm in diameter) for your classroom bulletin board showing the maximum number of regions for 5, 6, and 7 points.

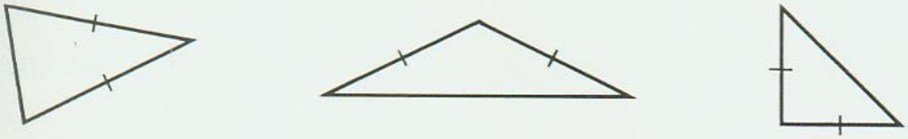
B.

3. Each triangle below is an equilateral triangle. Measure the angles. (Trace and extend the sides if necessary.) Copy and complete the generalization.



Generalization: If a triangle is an equilateral triangle, then it has $\underline{?}$ angles.

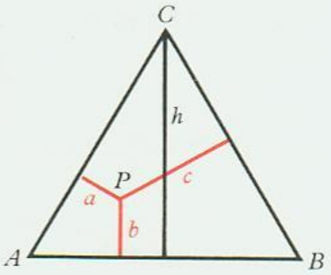
4. Each triangle below has two congruent sides. Trace the isosceles triangles and bisect the angles formed by the congruent sides. Observe the relationship between the bisector and the opposite side.



Generalization: In a triangle with two congruent sides, the bisector of the angle formed by the congruent sides is $\underline{?}$ to the third side.

C.

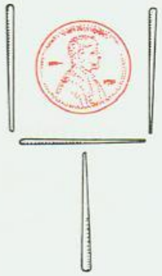
5. Trace $\triangle ABC$ with $AB = BC = AC$. Choose a point P inside the triangle and construct the perpendiculars from P to the sides of the triangle. Measure h, a, b, and c to the nearest mm. Try as many cases with different points P as needed to state a generalization.



PROBLEM SOLVING

The four toothpicks are arranged to represent a glass with a penny inside.

Move *exactly two* toothpicks so that a glass of the same size is formed and the penny (unmoved) is outside the glass.



2–2 False Generalizations and Counterexamples

SOMEONE TOLD ME THAT IF I PUT A PENNY IN MY SHOE IT WOULD BRING ME GOOD LUCK ... SO I TRIED IT..
...GOT THE BIGGEST BLISTER YOU EVER SAW !!

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This cartoon illustrates a situation in which a generalization, thought to be true, was dramatically shown to be false!

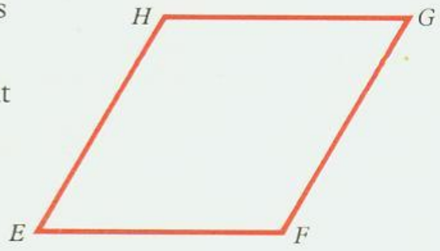
To show that a generalization is false, we often give a *counterexample*. The following three situations show how a counterexample exposes a false generalization.

Example 1

False Generalization: If a quadrilateral has four congruent sides, it has four congruent angles.

Comment: To show that the generalization is false, we must provide a quadrilateral with four congruent sides which does *not* have four congruent angles.

Counterexample: Figure *EFGH* has all sides congruent but $\angle E$ is not congruent to $\angle F$.

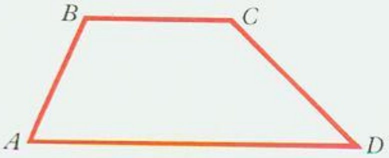


Example 2

False Generalization: If a quadrilateral has a pair of parallel sides, it has a pair of congruent sides.

Comment: To show that the generalization is false, we must produce a quadrilateral with a pair of parallel sides which does *not* have a pair of congruent sides.

Counterexample: Figure *ABCD* has $\overline{BC} \parallel \overline{AD}$ but no two sides are congruent.

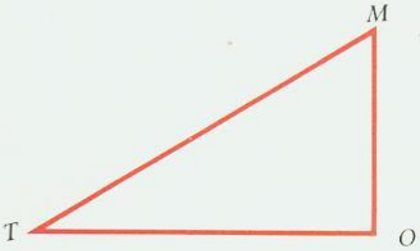


Example 3

False Generalization: If a triangle has a right angle, it has two congruent sides.

Comment: To show that the generalization is false, we must produce a triangle with a right angle which does *not* have two congruent sides.

Counterexample: Figure *TOM* has a right angle ($\angle O$) but all three sides have different lengths.



A counterexample can now be described as shown here.

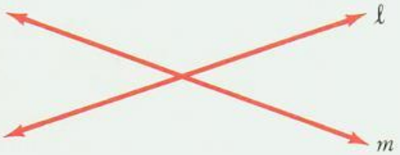
Definition 2-1

A **counterexample** is a single example that shows a generalization to be false.

Example 4

This figure is a counterexample for the following false generalization.

If two lines intersect, then they form right angles.



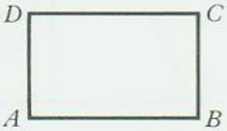
EXERCISES

A.

For exercises 1 and 2, a generalization based on the cases presented is given. Indicate whether you think the generalization is true or false. If it is false, give a counterexample.

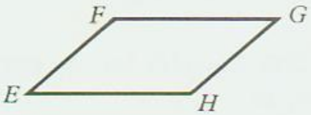
1.

Case 1



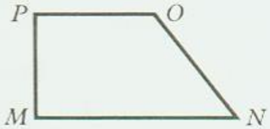
$\overline{AB} \parallel \overline{DC}$

Case 2



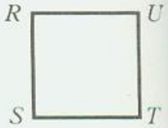
$\overline{EF} \parallel \overline{HG}$

Case 3



$\overline{MN} \parallel \overline{PO}$

Case 4

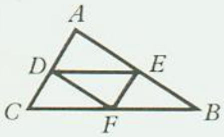


$\overline{RS} \parallel \overline{UT}$

Generalization: All quadrilaterals have a pair of parallel sides.

2.

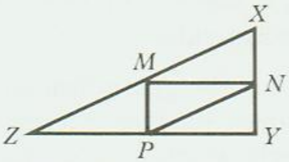
Case 1



D, E, F are midpoints

Perimeter of $\triangle ABC$ is twice perimeter of $\triangle DEF$.

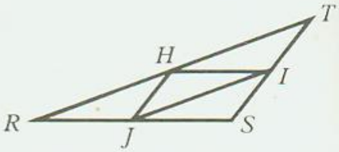
Case 2



M, N, P are midpoints

Perimeter of $\triangle XYZ$ is twice perimeter of $\triangle MNP$.

Case 3



H, I, J are midpoints

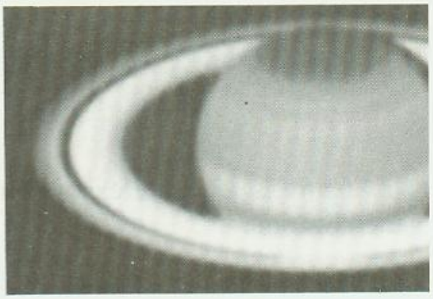
Perimeter of $\triangle RST$ is twice perimeter of $\triangle HIJ$.

Generalization: The perimeter of a triangle is twice the perimeter of the triangle formed by joining the midpoints of its sides.

Activity

Counterexamples play an important role in science. One purpose of space exploration programs is to confirm or disprove theories scientists have formed. Voyager's view of Saturn's rings gave surprising results. Before the Voyager space flights astronomers held many theories concerning Saturn's rings. Were any new findings counterexamples of any of these theories?

Bring to class or describe articles from science magazines or newspapers which support your views.



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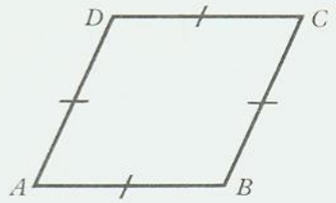
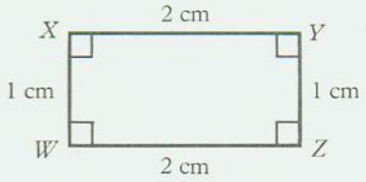


3. For which of the following statements would figure *ABCD* be a counterexample?
   a. When all sides of a quadrilateral are equal in length, all angles are equal in measure.
   b. When all angles of a quadrilateral are equal in measure, all sides are equal in length.
   c. When one pair of sides of a quadrilateral is congruent, a second pair of sides is also congruent.



4. For which of the following statements is polygon *WXYZ* a counterexample?
   a. A polygon with congruent sides is a regular polygon.
   b. A polygon with congruent angles is a regular polygon.
   c. A quadrilateral with congruent angles is a convex quadrilateral.

C.

In exercises 5 and 6 indicate whether the statement is true or false. If false, give a counterexample.

5. Given any $\triangle ABC$ the perpendicular bisector of $\overline{AB}$ intersects the perpendicular bisector of $\overline{BC}$ at a point inside the triangle.
6. Given any $\triangle ABC$, the line through A perpendicular to $\overleftrightarrow{BC}$ and the line through B perpendicular to $\overleftrightarrow{AC}$ intersect at a point inside the triangle.

PROBLEM SOLVING

This array of numbers, called *Pascal's triangle,* is named after the French mathematician Blaise Pascal (1623–1662).

1. Observe the relationship between each number and the numbers nearest it in the row above. Copy the array and complete at least three more rows.
2. Find the sum of the numbers in each row. State a generalization about these sums.
3. Find the polynomials for $(a + b)^2$, $(a + b)^3$, and $(a + b)^4$. List the coefficients of the terms in each polynomial. State a generalization about how this pattern relates to Pascal's Triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

2–3 Developing Geometry Using Deductive Reasoning

So far in our study, we have looked for objects in our world that suggest geometric ideas. We have chosen the most basic ideas—point, line, and plane—and called them **undefined terms**.

Using the undefined terms, we have provided **definitions** to describe other geometric figures, such as triangles, segments, and angles. We have also defined relationships, such as congruent, parallel, and perpendicular.

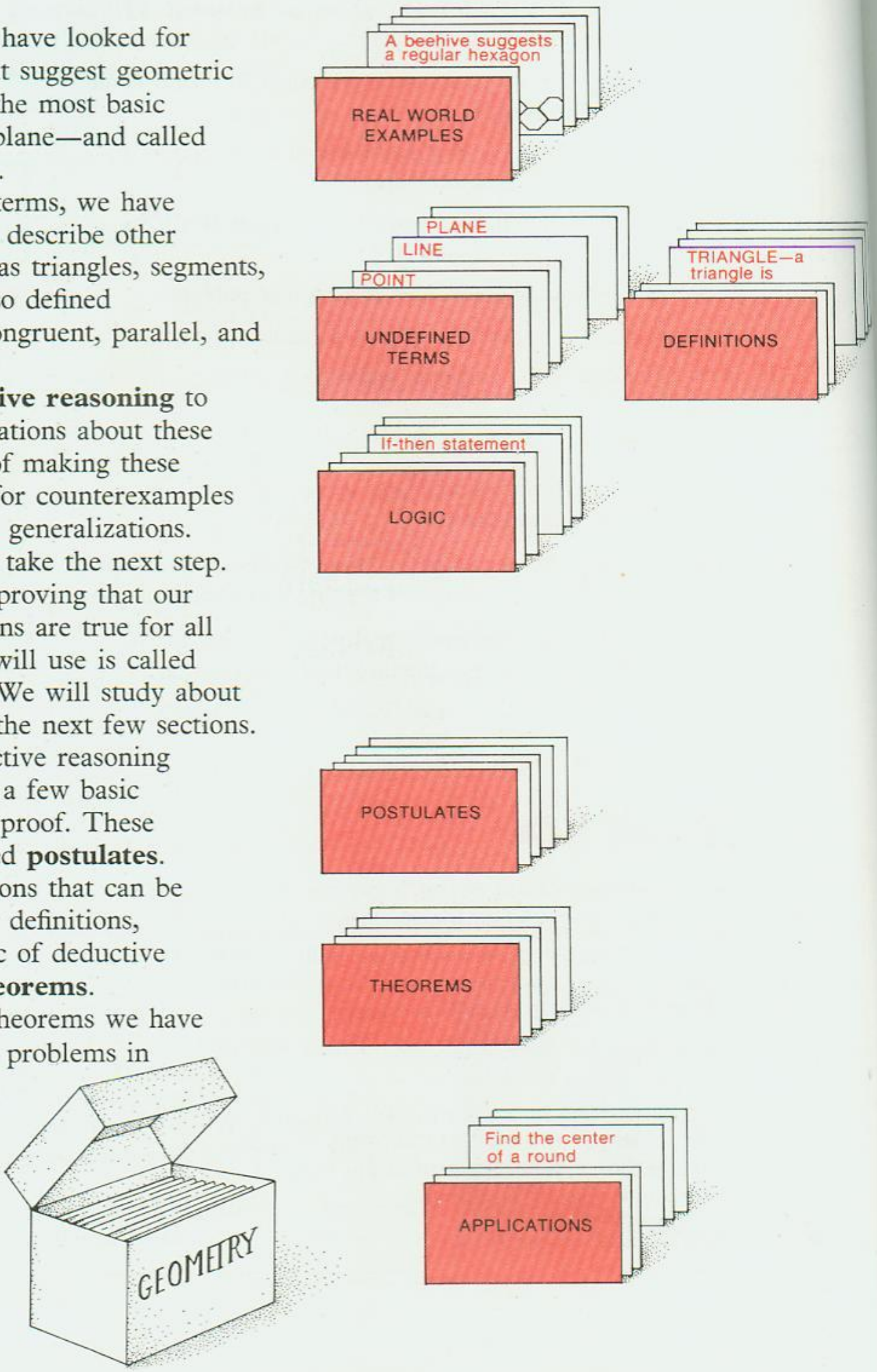
We then used **inductive reasoning** to discover some generalizations about these figures. In the process of making these discoveries, we looked for counterexamples that would disprove the generalizations.

We are now ready to take the next step. We need a method for proving that our discovered generalizations are true for all cases. The method we will use is called **deductive reasoning**. We will study about deductive reasoning in the next few sections.

The process of deductive reasoning requires that we accept a few basic generalizations without proof. These generalizations are called **postulates**.

All other generalizations that can be proved to be true using definitions, postulates, and the logic of deductive reasoning are called **theorems**.

Finally, we use the theorems we have proved to help us solve problems in everyday life.



PEANUTS
WHY SHOULD I GET RID OF MY BLANKET? WHY? GIVE ME A GOOD REASON!
IF YOU CAN GIVE ME ONE GOOD REASON WHY I SHOULD, I'LL GET RID OF IT!
BECAUSE IT MAKES YOU LOOK STUPID, THAT'S WHY! IT MAKES YOU LOOK STUPID, AND SILLY, AND FOOLISH AND IGNORANT!
I KNEW YOU COULDN'T THINK OF A GOOD REASON!!!
7-22
SCHULZ

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In the previous sections of this chapter we have used inductive reasoning to discover generalizations. Now we will explore logic and deductive reasoning and their role in the proof of theorems. The process of deductive reasoning involves three steps.

We outline these three steps for this theorem:

If two sides of a triangle are congruent,
then the two opposite angles are congruent.

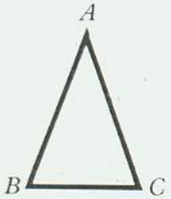
Deductive Reasoning

Step 1 Start with the given conditions (the hypothesis).

Step 2 Use logic and definitions, postulates, or previously proved theorems to justify a series of statements or steps which lead to the desired result.

Step 3 Assert the result (the conclusion).

Given: $\triangle ABC$ is a triangle with $\overline{AB} \cong \overline{AC}$.



The statements leading to the conclusion are not included here. In this chapter you will learn some reasoning patterns that will help you select, sequence, and give reasons for appropriate statements.

Therefore, $\angle B$ and $\angle C$ are congruent.

After we use logic to supply the correct statements in Step 2 of the sample proof above, we will have proved this theorem:

If *two sides of a triangle are congruent,*
hypothesis

then *the two opposite angles are congruent.*
conclusion

EXERCISES

A.

1. Webster's New Collegiate Dictionary gives this "definition" of a plane: A plane is a flat or level surface.

   List three words in the "definition" that should also be understood and for which one could try to write a definition.

2. Write a definition for one of the three terms in exercise 1. List any words in your definition that might also need to be defined.
3. Do you think the term *plane* should be selected as an undefined term? Explain.
4. Write a definition of *space* using the undefined terms set and point.
5. Write a statement that you think is a theorem in geometry.

Activity

One objective of a geometry course is to provide practice in logical reasoning in everyday life. In our daily lives we sometimes draw conclusions with little or no supporting evidence.

Accept this story as an accurate depiction of an event that actually happened:

Little Jack Horner sat in a corner
eating his Christmas pie.
He put in his thumb, pulled out a plum
and said "What a good boy am I."

Which of these conclusions have you accepted (perhaps without being aware of it) about this story?

1. Jack was eating a plum pie.
2. It was Christmas day.
3. Jack felt he was a "good boy" because he pulled out the plum.
4. Jack was sitting in the corner because he was being punished.
5. Jack was a child.

Now reread the story. Which of these conclusions are accurate, based only on the information given in the story?

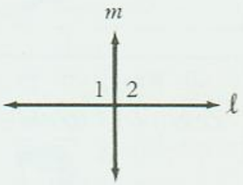
B.

In exercises 6–9, the given information and a conclusion based on deductive reasoning are stated. The statements leading from the hypothesis to the conclusion are omitted. State the theorem that was proved.

6. **Given:** Lines ℓ and m form a linear pair of angles, $\angle 1$ and $\angle 2$. $\angle 1 \cong \angle 2$

⋮

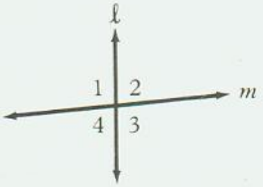
Therefore, $\ell \perp m$.



7. **Given:** $\angle 1$ and $\angle 3$ are vertical angles.

⋮

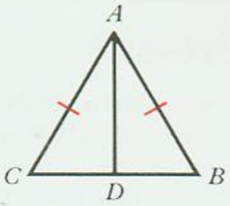
Therefore, $\angle 1 \cong \angle 3$.



8. **Given:** In isosceles $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$, $\overline{AD}$ bisects $\angle A$.

⋮

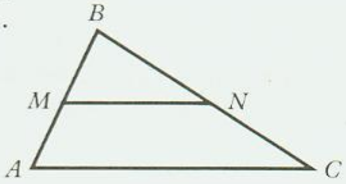
Therefore, $\overline{CD} \cong \overline{DB}$.



9. **Given:** $\overline{MN}$ joins the midpoints of $\overline{AB}$ and $\overline{BC}$ in $\triangle ABC$.

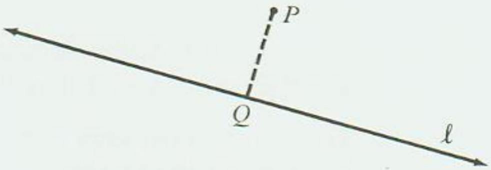
⋮

Therefore, $\overleftrightarrow{MN} \parallel \overleftrightarrow{AC}$.



C.

10. Write a logical argument to convince a friend that the shortest distance from a point P to a line ℓ is the length of $\overline{PQ}$ where Q is on ℓ and ℓ is perpendicular to $\overline{PQ}$.



PROBLEM SOLVING

Three ambassadors, all from different countries, paid \$30 for a suite of rooms at a hotel. Each contributed \$10. The clerk later realized that only \$25 should be charged, and gave the bellhop \$5 to return to them. The bellhop reasoned that their language differences might make it difficult for them to decide how to split up the \$5 refund, so the bellhop kept \$2 and returned \$3 to the ambassadors.

Each had paid \$10, minus the \$1 refund, or \$9 for the room. \$9 × 3 = \$27. \$27 plus the \$2 kept by the bellhop amounts to \$29! The original charge, however, was \$30. What happened to the other \$1?

2–4 Types of If-Then Statements

This cartoon illustrates a type of statement that is important in deductive reasoning. The examples below show statements in the "if-then" form.

If p, then q.

If it snows today, then we will ski.
If an isosceles triangle has a 60° angle, then it is an equilateral triangle.
If Lucy does not laugh, then Snoopy is sad.

PEANUTS
IF YOU THINK THOSE ARE FUNNY FACES YOU'RE MAKING, THEN YOU'RE SADLY MISTAKEN!
NOBODY APPRECIATES GOOD HUMOR ANY MORE
SCHULZ
6-14

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Definition 2–2 describes this type of statement. The examples below present some skills needed in order to work with and understand if-then statements.

Definition 2–2

An **if-then statement** is a statement of the form—if p, then q—where p and q are simple statements. p is called the **hypothesis**. q is called the **conclusion**. The symbol $p \rightarrow q$ (read p *implies* q) is used to represent an if-then statement.

Example 1 *Given an if-then statement, state the hypothesis and conclusion.*
If the weatherman says it will rain today, then we will not play tennis.

Hypothesis (p): The weatherman says that it will rain today.
Conclusion (q): We will not play tennis.

Example 2 *Given the hypothesis and conclusion, write the if-then statement.*

Hypothesis (p): Figure $ABCD$ is a square.
Conclusion (q): It has four congruent sides.
If-then ($p \rightarrow q$): If figure $ABCD$ is a square, then it has four congruent sides.

Example 3 *Given a statement in another form, write it as an if-then statement.*

The temperature is below 0°C when it snows.
If-then ($p \rightarrow q$): If it snows, then the temperature is below 0°C.

To learn more about the process of deductive reasoning we first must be able to decide whether an if-then statement—like the one shown here—is true. (Is it true? Explain.)

IF FISH CAN SWIM THEN GIRAFFES CAN FLY.
IS THIS TRUE OR FALSE ?

Now consider the following example. Suppose a teacher makes the if-then statement below. For which of the cases do you feel you were treated unfairly and the teacher didn't tell the truth? Your answer to this question will help you see when an if-then statement is false.

If you get A's on all Geometry tests, then you will get an A in the course.

Case 1	You get A's on all tests. (hypothesis true)	You get an A in the course. (conclusion true)
Case 2	You get A's on all tests. (hypothesis true)	You don't get an A in the course. (conclusion false)
Case 3	You don't get A's on all tests. (hypothesis false)	You get an A in the course. (conclusion true)
Case 4	You don't get A's on all tests. (hypothesis false)	You don't get an A in the course. (conclusion false)

The only time you were treated unfairly is in Case 2.

The results of the example above are summarized in the descriptions of true and false if-then statements given here.

An if-then statement is true if whenever the hypothesis is true, the conclusion is also true. Or, saying it another way, an if-then statement is false only when the hypothesis is true and the conclusion is false.

EXERCISES

A.

In exercises 1–8, first state the hypothesis (p) and the conclusion (q) of the if-then statement. Then decide whether or not the statement is true.

1. If Jana is 15 years old, then Jana is too young to vote in United States elections.
2. If a ball is tossed up, then it will come down.
3. If some apples are red, then horses have four legs.
4. If Rodney bought a pear, then Rodney bought a vegetable.
5. If two lines intersect, then those two lines are not parallel.
6. If three lines are concurrent, then the lines are parallel.
7. If $\triangle ABC$ is isosceles, then $\triangle ABC$ is equilateral.
8. If $\triangle ABC$ is equilateral, then $\triangle ABC$ is isosceles.

In exercises 9–14 write if-then statements ($p \to q$) with the given hypothesis (p) and conclusion (q).

9. Hypothesis (p): A man lives in Denver.
   Conclusion (q): He lives in Colorado.
10. Hypothesis (p): Two lines intersect to form right angles.
    Conclusion (q): The lines are perpendicular.
11. Hypothesis (p): Two lines are perpendicular.
    Conclusion (q): They intersect to form congruent right angles.
12. Hypothesis (p): A number is prime.
    Conclusion (q): The number has exactly two divisors.
13. p: Two lines are parallel.
    q: They do not intersect.
14. p: Two angles are congruent.
    q: They have the same measure.

Activity

Find statements from magazines or newspapers that involve if-then statements or statements that can be rewritten as if-then statements without changing the meaning. Discuss each statement and the logic behind it.

B.

In exercises 15–24, identify the hypothesis and conclusion and rewrite the statement in if-then form without changing its meaning.

15. A number is even if it ends in 2.
16. Two lines are perpendicular if they meet at right angles.
17. An equiangular triangle must be equilateral.
18. A number is odd whenever it ends in 5.
19. Do as I say and you will be rich.
20. You will improve provided that you work hard.
21. A triangle is isosceles whenever two of its angles are congruent.
22. A line which bisects a segment contains the midpoint of the segment.
23. B is the midpoint of $\overline{AC}$ provided that $\overline{AB} \cong \overline{BC}$.
24. All triangles are polygons.

C.

25. Write as many if-then statements as you can that have the same meaning as the statements in this advertisement from the Nowork Company.

    > Join Nowork Company and make a fast buck. Work for Nowork and get promoted fast. We hire only smart people who are too dumb to know they are smart. Nowork assures you excellent training at no pay. You will like working for Nowork Company. We hire any human. We fire any human.

26. Write a true if-then statement (if a, then b), and then rewrite it in each of these forms: b if a; a only if b; a is all that is needed to insure b; b must be true in order for a to be true. Which do you think are true?

PROBLEM SOLVING

Four adults and two children must cross a river in a canoe that will hold only one adult or two children at one time. It will not carry one adult and one child. Explain how the six people can get across the river. What is the least number of times the canoe must cross?

2–5 Converse, Inverse, and Contrapositive

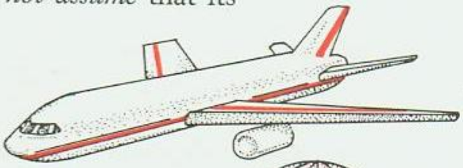
If we begin with a true if-then statement, we can form three types of related statements called the *converse,* the *inverse,* and the *contrapositive* of the original statement. Each type is described below and is compared with the original if-then statement.

$\sim$p
means "not p."

Statement:
$p \rightarrow q$
If a flag is the U.S. flag,
then it has stars.

Converse of the statement:
$q \rightarrow p$
If a flag has stars, then it
is the U.S. flag.

When an if-then statement $p \rightarrow q$ is true, we *should not assume* that its converse $q \rightarrow p$ is necessarily true.



Statement:
$p \rightarrow q$
If a vehicle is an airplane, then
the vehicle is built to fly.

Inverse of the statement:
$\sim p \rightarrow \sim q$
If a vehicle is not an airplane, then
the vehicle is not built to fly.

When an if-then statement $p \rightarrow q$ is true, we *should not assume* that its inverse $\sim p \rightarrow \sim q$ is necessarily true.

Statement:
$p \rightarrow q$
If you live in Chicago, then
you live in Illinois.

Contrapositive of the statement:
$\sim q \rightarrow \sim p$
If you do not live in Illinois, then
you do not live in Chicago.

When an if-then statement $p \rightarrow q$ is true we *can assume* that its contrapositive $\sim q \rightarrow \sim p$ is also true.

We summarize these results as follows:

$p \to q$	given statement	If p, then q	true (assumed)
$q \to p$	converse	If q, then p	not necessarily true
$\sim p \to \sim q$	inverse	If not p, then not q	not necessarily true
$\sim q \to \sim p$	contrapositive	If not q, then not p	always true

When a given statement and its converse are both true we can combine the two statements into one statement by using the phrase *if and only if.*

Statement:
$p \to q$ If today is Tuesday, then tomorrow is Wednesday.

Converse of the statement: If tomorrow is Wednesday, then today is Tuesday.
$q \to p$

if and only if statement: Today is Tuesday if and only if tomorrow is Wednesday.
$p \leftrightarrow q$

Every definition can be stated using an *if and only if* statement. For example, consider this definition.

An equilateral triangle is a triangle with all sides congruent to one another.

It can be restated like this.

A figure is an equilateral triangle if and only if it is a triangle with all sides congruent to one another.

The restatement actually combines two true statements:

1. If a triangle is equilateral, then it has all sides congruent.
2. If a triangle has all sides congruent, then it is equilateral.

We summarize this idea as follows:

p if and only if q means the same thing as *if p, then q* **and** *if q, then p.* When this statement is true we say p and q are *equivalent statements.*

EXERCISES

A.

For exercises 1–4, write the *converse* of each statement. Rewrite statements in if-then form if necessary. For which exercises are the statement and converse both true?

1. If a person is swimming, then that person is wet.

2. If two lines are perpendicular, then they intersect.

3. If a person is poor, then that person doesn't have much money.

4. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

For exercises 5–8 write the *inverse* of each statement.

5. If a person steals, then the person is dishonest.

6. If a figure is a triangle, then it is a polygon.

7. Any team that wins four World Series games wins the series.

8. Two planes are parallel if they do not intersect.

For exercises 9–12 write the *contrapositive* of each statement.

9. If you drive a car legally, then you are 16 years old or older.

10. All right angles are congruent.

11. We will win the championship if we win the game tonight.

12. If a triangle is equilateral, then it is equiangular.

Activity

Let's Analyze an Ad

ABC
The vitamin made especially for you!
ABC
VITAMINS

(1) If you want to feel your very best, take one ABC tablet each day. (2) People who take ABC vitamins care about their health. You owe it to those close to you to take care of yourself. Start taking ABC today!

When we hear statements like (1) we often erroneously assume the *converse*. "If you take an ABC tablet daily, then you'll feel your very best." When we hear statements like (2) we often erroneously assume the *inverse:* "If I don't take ABC vitamins, then I don't care about my health."

Copy a TV or magazine ad. What are some possible erroneous assumptions you might make?

B.

In exercises 13–18 write two simple statements that are equivalent.

13. A triangle is equilateral if and only if it is equiangular.
14. An angle is a right angle if and only if its measure is 90°.
15. Two lines in a plane are parallel if and only if they have no points in common.
16. A quadrilateral is a rectangle if and only if it has four right angles.
17. A quadrilateral is a parallelogram if and only if it has two pairs of parallel sides.
18. Two angles are complementary if and only if the sum of their measures is 90°.

C.

19. Write the converse, inverse, and contrapositive of these sentences of the form $p \rightarrow q$.

Make a chart like the following and then mark each entry true or false.

	Statement	Converse	Inverse	Contrapositive
If I live in Atlanta, I live in Georgia.	T	F		
If a number is positive, then it is greater than 6.	F	T		
If two segments have the same length, then they are congruent.				
If a figure is a regular polygon, then all sides are congruent.				

PROBLEM SOLVING

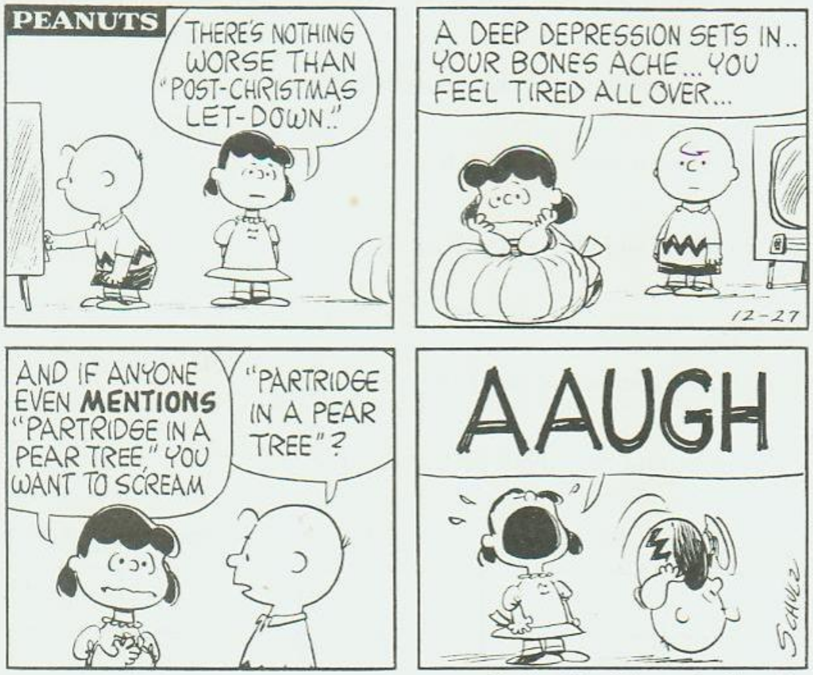
Two women, Alice and Carol, and two men, Brian and David, are athletes. One is a swimmer, a second is a skater, a third is a gymnast, and a fourth is a tennis player. One day they were seated around a square table:

1. The swimmer sat on Alice's left.
2. The gymnast sat across from Brian.
3. Carol and David sat next to each other.
4. A woman sat on the skater's left.

Who is the tennis player?

2–6 Patterns of Reasoning

We will now study several patterns of reasoning. They will be used often when you prove theorems. Study the cartoon and discussion below.


PEANUTS
THERE'S NOTHING WORSE THAN "POST-CHRISTMAS LET-DOWN."
A DEEP DEPRESSION SETS IN.. YOUR BONES ACHE... YOU FEEL TIRED ALL OVER...
12-27
AND IF ANYONE EVEN MENTIONS "PARTRIDGE IN A PEAR TREE," YOU WANT TO SCREAM
"PARTRIDGE IN A PEAR TREE"?
AAUGH
SCHULZ

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The first pattern of reasoning we will study begins with a true if-then statement. The hypothesis is established as true. We can then conclude that the conclusion is also true.

$p \to q$ is true.	If anyone mentions partridge in a pear tree, then Lucy will scream.
p is given.	Charlie says "partridge in a pear tree."
Conclude that q is true.	Therefore, Lucy screams.

The above example illustrates a pattern of reasoning called *affirming the hypothesis*. It is based on establishing that the hypothesis of a true if-then statement is also true. It is also named with the latin words *modus ponens*.

Definition 2-3

Affirming the hypothesis is a pattern of reasoning represented as follows:
Whenever $p \to q$ is true and p is true, we can conclude q is true.

A second pattern of reasoning is the basis for indirect proof which you will use in later chapters. In the cartoon, the pattern is:

$p \to q$ is true.
$\sim q$ (not q is given).

Conclude that $\sim p$ is true
(p is false).

If you really loved me, then you'd stop playing the piano.
Schroeder didn't stop.

Therefore, Schroeder doesn't love Lucy."

This pattern of reasoning is called *denying the conclusion*. It is also named by the latin words *modus tollens*.

The next cartoon suggests a third pattern that is used in writing proofs. It ties together a sequence of statements in the proof of a theorem. The pattern is:

$p \to q$ is true.
$q \to r$ is true.

Conclude that $p \to r$ is true.

Charlie Brown's "chain of reasoning":

If you have a broken heart, then the jagged edges poke you in the side.
If the jagged edges poke you in the side, then you can't sleep at night.

Therefore, if you have a broken heart, then you can't sleep at night.

This example illustrates a pattern of reasoning called the chain rule. It provides a way to "link" if-then statements in writing proofs.

PEANUTS
IF YOU REALLY LOVED ME, YOU'D STOP PLAYING THAT PIANO AND TALK TO ME..
THAT'S CALLED ANSWERING WITHOUT ANSWERING

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Definition 2-4

Denying the conclusion is a pattern of reasoning represented as follows:

Whenever $p \to q$ is true
and $\sim q$ (q is false).

We conclude: $\sim p$ (p is false).

PEANUTS
HOW CAN YOU TELL IF YOU HAVE A BROKEN HEART?
WELL, FOR ONE THING, IF YOU HAVE A BROKEN HEART, YOU CAN'T SLEEP AT NIGHT..
WHEN YOU ROLL OVER IN BED, THE JAGGED EDGES POKE YOU IN THE SIDE
I'M GLAD I TALKED TO AN EXPERT

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Definition 2-5

The **chain rule** is a pattern of reasoning represented as follows:

Whenever $p \to q$ is true
and $q \to r$ is true.

We conclude $p \to r$ is true.

EXERCISES

A.

For exercises 1–6 state the correct conclusion:

1. $p \to q$: If Julia votes today, she will vote for Fred Friendly.
   p: Julia voted today.
   Therefore q: ?

2. $p \to q$: If $\triangle ABC$ is isosceles, then it has two congruent angles.
   p: $\triangle ABC$ is isosceles.
   Therefore q: ?

3. $p \to q$: If it snows the temperature is below 0°C.
   $\sim q$: The temperature is at or above 0°C.
   Therefore $\sim p$: ?

4. $p \to q$: If a figure is an equilateral triangle, then it has all sides congruent and all angles congruent.
   $q \to r$: If a figure has all sides congruent and all angles congruent, then it is a regular polygon.
   Therefore $p \to r$: ?

5. $p \to q$: If two lines are parallel they have no points in common.
   $\sim q$: Lines m and n have a point in common.
   Therefore $\sim p$: ?

6. $p \to q$: Two perpendicular lines intersect.
   $q \to r$: Lines are not parallel if they intersect.
   Therefore $p \to r$: ?

Activity

Consider the following examples of reasoning often found in advertising. Decide if the companies used correct or incorrect reasoning patterns.

1. "Send your money in before October 1 and buy at $\frac{1}{2}$ price." Paula Joseph mailed her order on October 10. The company honored her request. Did the company lie?
2. "Send your money in before December 1 and buy at $\frac{1}{2}$ price." Angelo Silva sent his money on November 24, but the company refused his request. Did the company lie?
3. "This offer is good for people living in the continental United States." Namoonie Nakak lives in the Arctic. The company refused her offer. Was the company justified?
4. "You will be handsome if you use Neato." Jason does not use Neato but he is very handsome. Was the company wrong?
5. See if you can find and identify examples of correct and incorrect reasoning patterns in advertising.

B.

In exercises 7–9 fill in the missing information so that a correct reasoning pattern is given.

7. (Accept) A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.
(Given) ?
(Conclusion) Point C is equidistant from the endpoints of $\overline{AB}$.

8. (Accept) All horizontal lines are parallel.
(Given) ?
(Conclusion) $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are not horizontal lines.

9. (Accept) ?
(Given) $\angle ABC$ has measure greater than $90°$.
(Conclusion) $\angle ABC$ is an obtuse angle.

C.

In exercises 10–11 write the missing statement. (You should be able to do these exercises even if you don't know the meaning of all the terms used.)

10. **Theorem:** If $ABCD$ is a square, then $ABCD$ is not a kite.
Proof: 1. If $ABCD$ is a square, then $ABCD$ is a rhombus.
2. (missing statement)
3. If $ABCD$ is a parallelogram, then $ABCD$ is not a kite.
Therefore, if $ABCD$ is a square, $ABCD$ is not a kite.

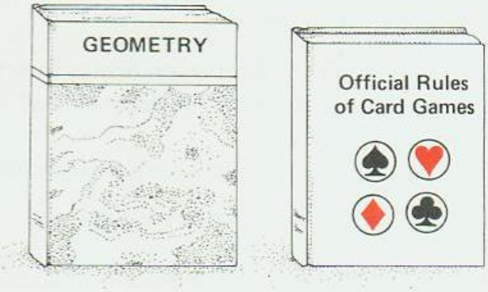
11. **Theorem:** If $\triangle ABC$ is a right triangle with $\angle C$ a right angle, then $\angle A$ and $\angle B$ are complementary.
Proof: 1. (missing statement)
2. If $m\angle A + m\angle B + m\angle C = 180$ and $m\angle C = 90$, then $m\angle A + m\angle B = 90$.
3. (missing statement)
Therefore, if $\triangle ABC$ is a right triangle with $\angle C$ a right angle, then $\angle A$ and $\angle B$ are complementary.

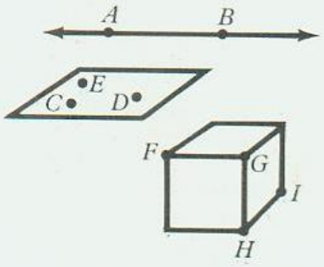
PROBLEM SOLVING

Three people named Lee, Martinelli, and Nielsen fill the positions of accountant, cashier, and clerk in a department store. If Nielsen is the cashier, Martinelli is the clerk. If Nielsen is the clerk, Martinelli is the accountant. If Martinelli is not the cashier, Lee is the clerk. If Lee is the accountant, Nielsen is the clerk. What is each person's job?

2–7 Postulates in Geometry

The postulates of geometry are very important in the process of deductive reasoning. They can be compared to the rules of a game. In the "game of geometry" we accept the postulates as true and use them to help us prove theorems.

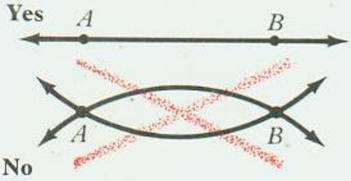

GEOMETRY
Official Rules of Card Games
♠ ♥ ♦ ♣



To assure that points exist we accept this postulate. The postulate also gives information about lines and planes.

Points Existence Postulate

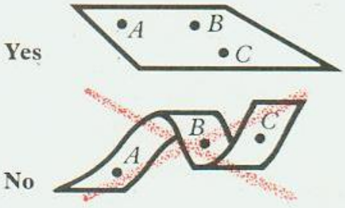
Space exists and contains at least four noncoplanar points. A plane contains at least three noncollinear points. A line contains at least two points.



To assure that a line is *straight,* we want one and only one line to contain any two points. We can also say *two points determine a line.*

Point-Line Postulate

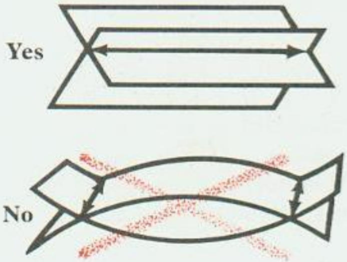
Two points are contained in one and only one line.



To assure that a plane does not twist and turn in space, we want one and only one plane to contain any three noncollinear points. We can also say *three noncollinear points determine a plane.*

Point-Plane Postulate

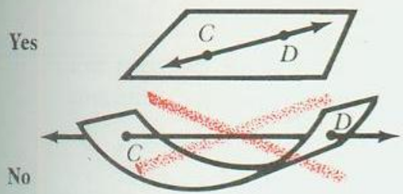
Three noncollinear points are contained in one and only one plane.



To assure that a plane is straight, we want two planes to intersect in just one line, not two.

Plane Intersection Postulate

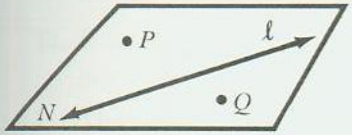
If two planes intersect, then they intersect in exactly one line.



To assure that a plane is flat, we want a plane to contain all points of a line if we know that it contains two points of the line.

Two Points, Line, Plane Postulate

If two points are in a plane, then the line containing them is in the plane.



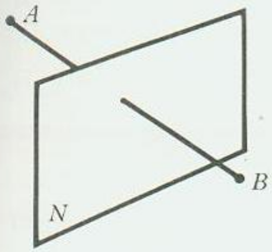
We want a line to separate a plane into two half-planes. We can use this postulate to decide whether two points lie on the same side of a line or on opposite sides of the line.

Plane Separation Postulate

Let N be a plane and ℓ a line in N. The points of the plane not on ℓ form two half-planes such that

a. each half-plane is a convex set.

b. if P is in one half-plane and Q is in the other, then $\overline{PQ}$ intersects ℓ.



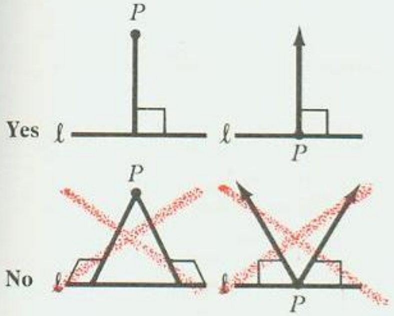
We want a plane to separate space into half-spaces. We can use this postulate to decide whether two points lie on the same side of a plane or on opposite sides of a plane.

Space Separation Postulate

Let N be a plane in space. The points of space not on N form two half-spaces such that

a. each half-space is a convex set.

b. if a point A is in one half-space and B is in the other, $\overline{AB}$ intersects N.



We want to have *only one* line through a given point that is perpendicular to a given line or plane.

Perpendicular Postulate

Given a point and a line in a plane, there is exactly one line through the point perpendicular to the given line. Given a plane in space and a point not in that plane, there is exactly one line through the point perpendicular to the given plane.

These are some of the postulates accepted in geometry. Other important postulates will be presented in the next lesson and in later chapters.

EXERCISES

A.

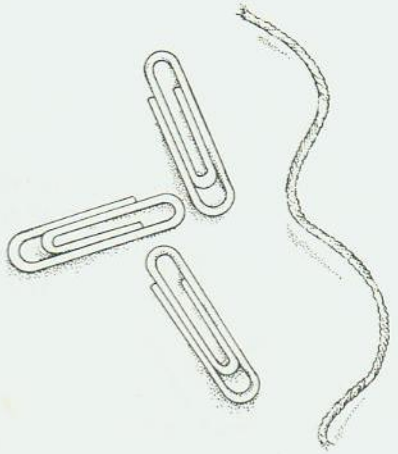
Complete exercises 1–8 using the words *point, line, plane,* or *space*. Name the postulate that suggests the completed statement.

1. If the two points are in a plane, then the _?_ that contains them is in the plane.
2. A _?_ contains at least three noncollinear points.
3. Two points are contained in one and only one _?_.
4. If two planes intersect they intersect in exactly one _?_.
5. There is exactly one _?_ through a given point and perpendicular to a given plane.
6. A plane separates _?_ into two half-spaces.
7. In a plane, there is exactly one _?_ through a given point and perpendicular to a given line.
8. A line separates a _?_ into two half-planes.

Activity

Use string and paper clips to make a model for these "postulates." What is the minimum number of pieces of string and paper clips needed to fulfill all the conditions of the postulates?

1. There is at least one piece of string.
2. There are exactly three paper clips on every piece of string.
3. Not all paper clips are on the same string.
4. There is exactly one string through any two paper clips.
5. Any two strings have at least one paper clip in common.



For exercises 9–16, state the postulate that allows you to conclude that the statement is true.

9. Two distinct planes M and N cannot both contain two distinct lines ℓ and m.

10. Two distinct lines ℓ and m cannot both contain two distinct points A and B.

11. Given a point P and a line ℓ in a plane, there cannot be two distinct lines through P and perpendicular to ℓ.

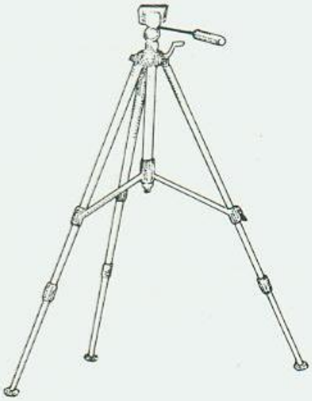
12. If points J and K are in different half-planes determined by a line ℓ, $\overline{JK}$ intersects ℓ.

13. Three noncollinear points A, B, and C cannot all be contained in both of the two distinct planes N and M.

14. If A and B are points in plane M, then $\overleftrightarrow{AB}$ has no points not in plane M.

15. Given a point P and a plane M, there cannot be two lines through P and perpendicular to plane M.

16. Space may contain more, but no less than four noncollinear, noncoplanar points.



17. Photographers and engineers use tripods to mount their cameras and other equipment. Explain why three legs are used instead of four. Which postulate applies to this situation?

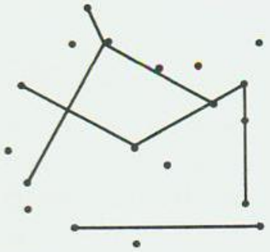
18. How many lines can be drawn through four noncollinear points in a plane? Five noncollinear points? Six? n noncollinear points?

19. How many planes are determined by four points, no three of which are collinear?

PROBLEM SOLVING

These 19 points are the corners of four shapes—a square, an envelope with a flap, a box, and an equilateral triangle. (Several edges have been drawn already.) No point is the corner of more than one shape but some points are on the edge of another shape.

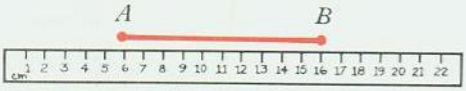
Trace this figure and draw in all four shapes.



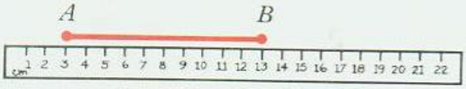
2–8 Some Measurement Postulates

In CB language, the "ten-twenty" (location) in the first picture is the 62 kilometer marker. The location in the second picture, is the 97 kilometer marker. How far has the motorist traveled?

In an earlier chapter (page 20) the idea of measurement of length was introduced intuitively and used freely to form definitions and draw conclusions. Here we look briefly at the postulate on which this idea of length is based.



$|6 - 16| = |16 - 6| = 10$



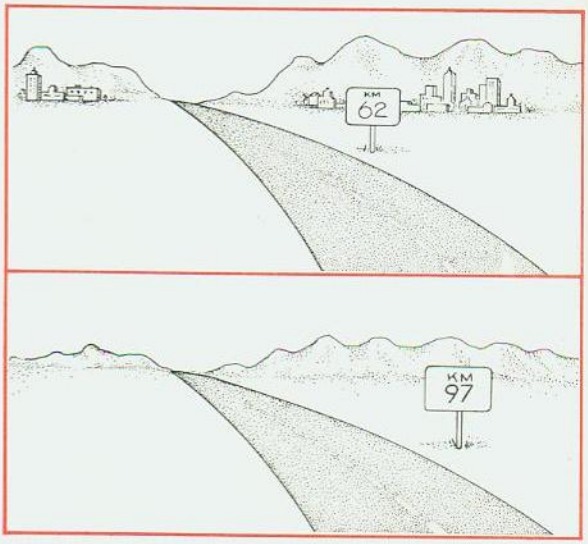
$|3 - 13| = |13 - 3| = 10$

These examples suggest that no matter where the ruler is placed, the length of the segment is the absolute value of the difference of the two numbers matched with the endpoints. The Ruler Postulate includes this idea.

Example 1 Find AB, BC, and AC.

$$AB = |3 - (-2)| = 5$$
$$BC = |18 - 3| = 15$$
$$AC = |-2 - 18| = 20$$

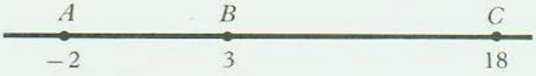
In the example, B is between A and C. Notice that the distance from A to B together with the distance from B to C is the same as the distance from A to C.
We state a definition.



The Ruler Postulate

a. To every pair of points there corresponds a unique positive number called the **distance** between the points.

b. The points on a line can be matched one-to-one with the real numbers so that the distance between any two points is the absolute value of the difference of their associated numbers.



Definition 2–6

Point B is **between** A and C if and only if A, B, and C are collinear and $AB + BC = AC$

A navigator for an airplane checked the course of the airplane (red lines in Figure 1 highlight the course, which is the angle between due north and the plane's line of motion). Later the course was changed as shown in Figure 2. By how many degrees had the course changed?

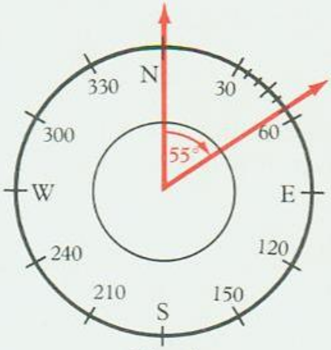


Fig. 1

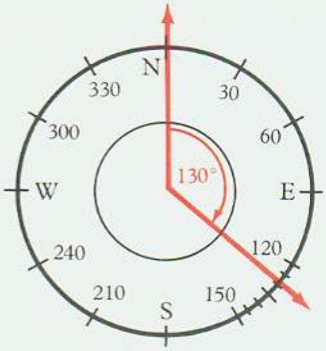
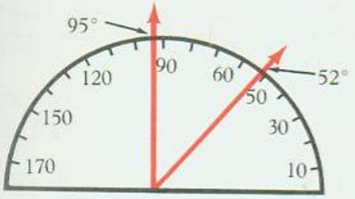


Fig. 2

In the first chapter (page 20) the idea of angle measure was introduced intuitively and used freely to make definitions and draw conclusions. Here we look briefly at the postulate on which this idea is based.



$|95 - 52| = |52 - 95| = 43$

If the protractor is turned so that the rays are matched with different numbers, the absolute value of the difference of the two numbers would be the same. The postulate includes this idea.

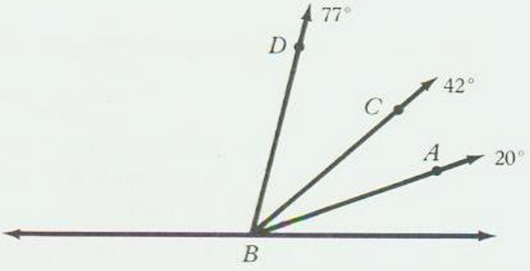
The Protractor Postulate

a. To each angle there corresponds a unique real number between 0 and 180 called the measure of the angle.

b. Let P be a point on the edge of half-plane H. Each ray in the half-plane or its edge with vertex P can be matched one-to-one with the real numbers n, $0 < n < 180$, so that the measure of an angle formed by a pair of noncollinear rays with vertex P is the absolute value of the difference of their associated numbers.

Example 2 Find $m\angle ABC$, $m\angle CBD$, and $m\angle ABD$.

$$m\angle ABC = |42 - 20| = 22$$
$$m\angle CBD = |77 - 42| = 35$$
$$m\angle ABD = |77 - 20| = 57$$



In the example $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BD}$. Notice that the measures of $\angle ABC$ and $\angle CBD$ taken together equal the measure of $\angle ABD$. We state a definition.

Definition 2-7

$\overrightarrow{BC}$ is **between** $\overrightarrow{BA}$ and $\overrightarrow{BD}$ if and only if $\overrightarrow{BC}$, $\overrightarrow{BA}$ and $\overrightarrow{BD}$ are coplanar and $m\angle ABC + m\angle CBD = m\angle ABD$.

EXERCISES

A.

A number associated with a point is called the *coordinate of the point*. Find the distance between a pair of points with the coordinates given in exercises 1–12.

1. 5, 2 **2.** 14, 8 **3.** 3, 12 **4.** 17, 0

5. 13, 21 **6.** -5, 13 **7.** 9, -2 **8.** -3, 15

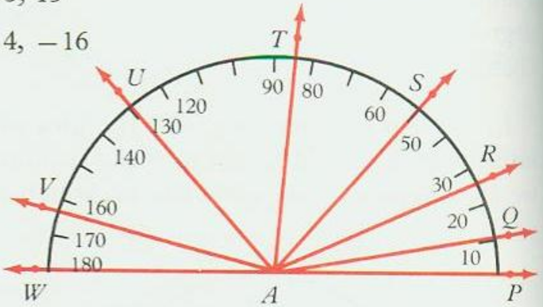
9. -12, 7 **10.** -3, -7 **11.** 0, -8 **12.** -4, -16

Give the measure of each angle in exercises 13–18.

13. $\angle QAS$ **14.** $\angle UAR$

15. $\angle QAV$ **16.** $\angle WAQ$

17. $\angle SAU$ **18.** $\angle VAR$



(Exs. 13–18)

B.

The points A, B, and C are collinear. The lengths of certain segments are given. Which point is between the other two?

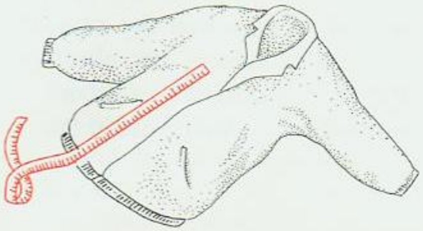
19. $AC = 4$, $CB = 8$, $AB = 12$

20. $BA = 9$, $BC = 12$, $AC = 3$

21. $CA = 20$, $BA = 11.6$, $CB = 8.4$

22. $AC = 9.3$, $CB = 6.5$, $AB = 15.8$

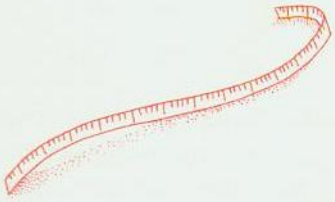
23. A jacket is going to have three buttonholes four inches apart. A tape measure is placed along the material. The buttonholes are marked by the numbers 7, 11 and 15. Which postulate or definition is being used?



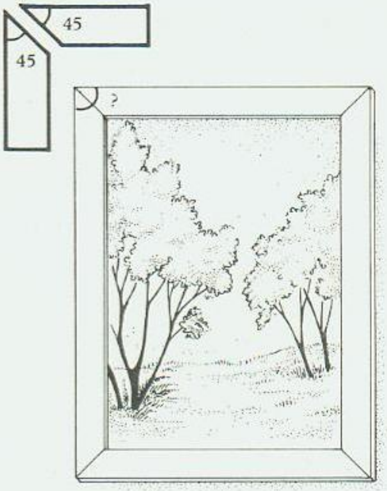
Activity

Estimate the length of these objects to the nearest metric unit indicated. Find the sum of the differences between your estimate and the actual length. The person with the smallest sum wins!

1. Distance around your waist in centimeters (use string)
2. Width of a penny in millimeters
3. Length of room in meters
4. Length of your normal pace in centimeters
5. Height of door in meters



24. Two sides of a picture frame are glued together to form a corner. Each side is cut at a 45° angle. What is the measure of the outside corner? Which postulate or definition is being used?



C.

In exercises 25–29, A, B, and C are collinear points with coordinates a, b, and c.

25. If $a < b < c$, $c = 54$, $AC = 26$, and $BC = 19$, find a and b.

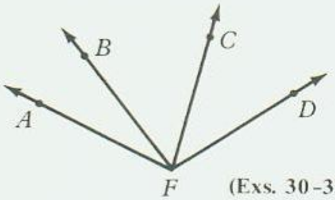
26. If $b = -7$, $c = 13$, and A is the midpoint of $\overline{BC}$, find a.

27. If $b = -10$, $c = 4$, and $BA = 28$, find BC and CA. Give two possible answers.

28. If C is between A and B, $BC = 8$, and $CA = 36$, find a, b, and c. Give two possible answers. How many answers are possible?

29. If C is the midpoint of $\overline{AB}$, $c = 14$, and $CB = 10$, give two possible coordinates for A.

Exercises 30–33 refer to an arrangement of rays in the same order as those in the figure.



(Exs. 30–33)

30. If $m\angle AFD = 120$ and $m\angle AFC = 2 \cdot m\angle CFD$, find $m\angle AFC$ and $m\angle CFD$.

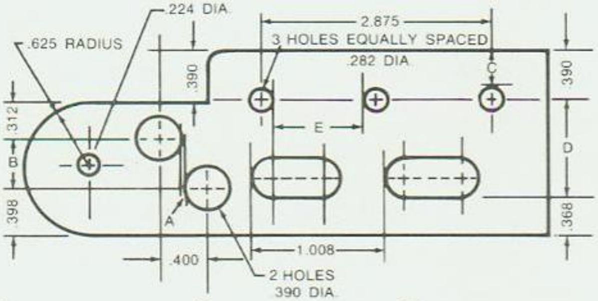
31. If $m\angle AFC = 3 \cdot m\angle AFB$, $\overrightarrow{FC}$ bisects $\angle AFD$, and $m\angle AFB = 16$, find $m\angle AFD$.

32. If $m\angle AFC = 2 \cdot m\angle CFD$, $m\angle AFB = \frac{1}{2}m\angle CFD$, and $m\angle AFD = 112$, find $m\angle AFB$, $m\angle BFC$, and $m\angle CFD$.

33. If $m\angle BFC = 3 \cdot m\angle AFB$, $m\angle CFD = 4 \cdot m\angle AFB$, and $m\angle AFD = (m\angle AFB)^2 - 240$, find the measure of $\angle AFB$.

PROBLEM SOLVING

A machinist is to make a metal plate like the one shown. The first step is to find the missing dimensions indicated by A, B, C, D, and E. Find them.



A = ____ B = ____ C = ____ D = ____ E = ____

Important Ideas—Chapter 2

Terms

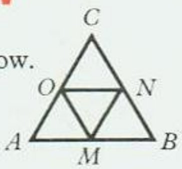
Generalization (p. 44)
Inductive reasoning (p. 45)
Counterexample (p. 49)
Undefined terms (p. 52)
Postulates (p. 52)
Theorems (p. 52)
Deductive reasoning (p. 53)
Hypothesis (p. 56)
Conclusion (p. 56)
If-then statement (p. 56)
Converse (p. 60)
Inverse (p. 60)
Contrapositive (p. 60)
Affirming the hypothesis (p. 64)
Denying the conclusion (p. 65)
Chain rule (p. 65)
Betweenness of a point (p. 72)
Betweenness of a ray (p. 73)

Postulates

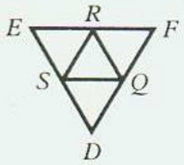
Points Existence Postulate (p. 68)
Point-Line Postulate (p. 68)
Point-Plane Postulate (p. 68)
Plane Intersection Postulate (p. 68)
Two Points, Line, Plane Postulate (p. 69)
Plane Separation Postulate (p. 69)
Space Separation Postulate (p. 69)
Perpendicular Postulate (p. 69)
Ruler Postulate (p. 72)
Protractor Postulate (p. 73)

Chapter 2—Review

1. Complete the generalization below.
   $\triangle ABC$ is equilateral.
   M, N, O are midpoints.
   $\triangle MNO$ is equilateral.



$\triangle EFD$ is equilateral.
R, Q, S are midpoints.
$\triangle RQS$ is equilateral.



Generalization: The midpoints of an _?_
form the vertices of an _?_.

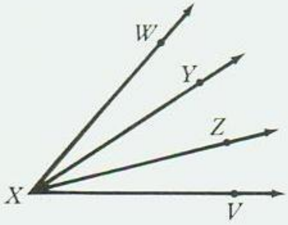
2. Produce counterexamples for the following false generalizations.
   a. If the diagonals of a quadrilateral are perpendicular, the quadrilateral is a square.
   b. If a quadrilateral has four congruent angles, it has four congruent sides.

3. What is the difference between a theorem and a postulate?

4. True-False
   a. Three points determine a plane.
   b. Given a line ℓ and a point P not on that line, there is only one line containing P that is perpendicular to ℓ.
   c. The intersection of a plane and a line may contain exactly two points.
   d. If A, B, and C are collinear, $AB = 10$ and $BC = 4$ then B is between A and C.

5. If S is between R and T, $ST = 6$, and $RS = 10$, find RT.

6. If $\angle WXY \cong \angle ZXY$, $m\angle WXY = 20$, and $m\angle YXV = 50$, find $m\angle ZXV$.



7. Identify the hypothesis and conclusion in the following:
   a. Two lines are parallel if they do not intersect
   b. All squares are rectangles.

8. Consider the statement:
   If a figure is a square, it has four right angles.
   a. Write the converse statement.
   b. Write the inverse statement.
   c. Write the contrapositive statement.
   d. Produce counterexamples for a and b.

In exercises 9–10 state a correct conclusion.

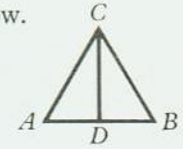
9. If a figure is a rectangle, the diagonals are congruent.
   $ABCD$ is a rectangle.
   Therefore, _?_.

10. If two lines are not in the same plane, they are not parallel.
    $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Therefore, _?_.

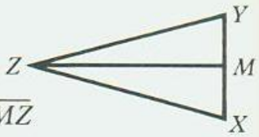
Chapter 2—Test

1. Complete the generalization below.

$\triangle ABC$ is isosceles and $\overline{CD}$ bisects $\angle C$.
What is the relationship of $\overline{CD}$ and $\overline{AB}$.



$\triangle XYZ$ is isosceles and $\overline{MZ}$ bisects angle Z.
What is the relationship of $\overline{MZ}$ and $\overline{XY}$?



Generalization: The bisector of the vertex angle of an isosceles triangle is $\underline{?}$ the base.

2. Produce a counterexample for the following false generalization. There is only one line perpendicular to a given line at a point on that line.

3. What is the purpose of having undefined terms?

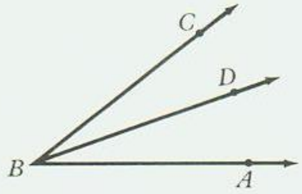
4. True-False

a. Two points determine a line.

b. A line separates space into two half-spaces.

c. If two planes intersect, the intersection is a point.

d. If X, Y, and Z are collinear and $XZ + ZY = XY$ then Z is between X and Y.



5. If B is between A and C, $AC = 10$, and $AB = 4$, find BC.

6. Fill in the blanks.

$m\angle ABD + \underline{?} = m\angle ABC.$

7. Identify the hypothesis and conclusion in the following statements:

a. A triangle is isosceles if it has two congruent sides.

b. All equilateral triangles have three congruent angles.

8. Consider the statement:
If two angles are right angles, they are congruent.

a. Write the converse statement.

b. Write the inverse statement.

c. Write the contrapositive statement.

d. Produce counterexamples to show a and b are false.

In exercises 9–10 state a correct conclusion.

9. If a triangle has three congruent angles, it is equilateral.
$\triangle ABC$ has three congruent angles.
Therefore, $\underline{?}$.

10. If two angles are right angles (90°), they are congruent.
$\angle A$ is not congruent to $\angle B$.
Therefore, $\underline{?}$.

Algebra Review

For each problem give the algebraic property illustrated.

A. Associative law (addition and multiplication)
B. Commutative law (addition and multiplication)
C. Distributive law (multiplication over addition)
D. Inverse property (addition and multiplication)
E. Identity property (addition and multiplication)
F. Addition/subtraction of equals
G. Multiplication/division of equals

1. $3 \cdot (7 \cdot 9) = (3 \cdot 7) \cdot 9$
2. If $x = 3$, then $7x = 21$
3. $7 + (-7) = 0$
4. If $x - 2 = 1$, then $x = 3$
5. $x \cdot (a + y) = xa + xy$
6. If $a = b$ then $a + c = b + c$
7. $a + 0 = a$
8. $a \cdot b = b \cdot a$

Solve each equation for x.

9. $x + 3 = 35$
10. $3x = 35$
11. $x - 7 = 19$
12. $\frac{x}{6} = 8$
13. $2x + 9 = 11$
14. $-4x = 8$
15. $10 - x = 42$
16. $\frac{x}{2} + 9 = -5$
17. $3x - 19 = 17$

Solve and graph on a number line.

18. $x > 3$
19. $7 < x$
20. $3x < 15$
21. $x + 1 \geq -17$
22. $x < \frac{4}{5}$
23. $x + 2 \geq 24$
24. $9 - x > 3$
25. $\frac{x}{2} > 10$
26. $3(x + 6) > 12$

Solve each equation.

27. $|x| = 9$
28. $3|x| = 21$
29. $|x + 1| = 4$
30. $|3x| = 21$
31. $|x| - 2 = 35$
32. $|3x + 2| = 17$
33. $|2x + 1| = 13$
34. $|x| - 9 = 25$
35. $3 - |x| = -5$

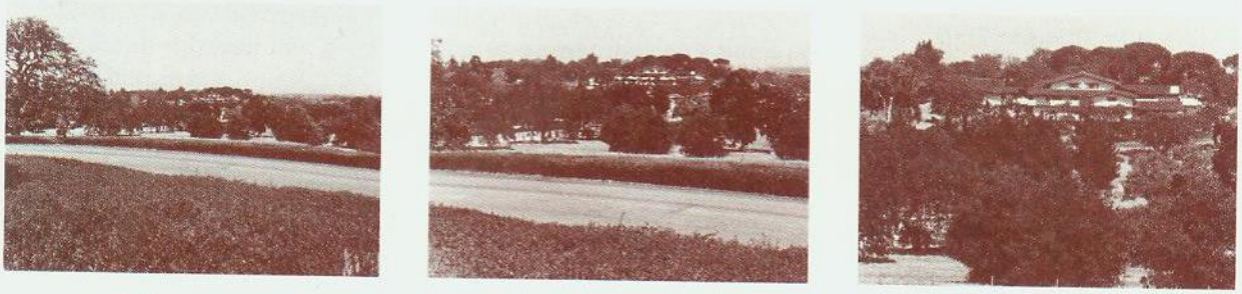
Evaluate.

36. $\sqrt{25}$
37. $\sqrt{121}$
38. $\sqrt{100 + 69}$
39. $\sqrt{121} + \sqrt{81}$
40. $\sqrt{4^3}$
41. $8\sqrt{9} - 6\sqrt{36}$

Photography: Lenses

Photographers often use different lenses, from wide angle to telephoto, to get the picture they want.

The three photos below are taken from the same position with the same camera. Each photo was taken with a different lens.



Wide angle

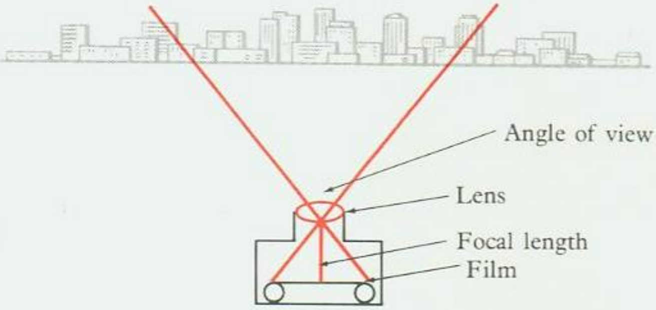
Normal

Telephoto

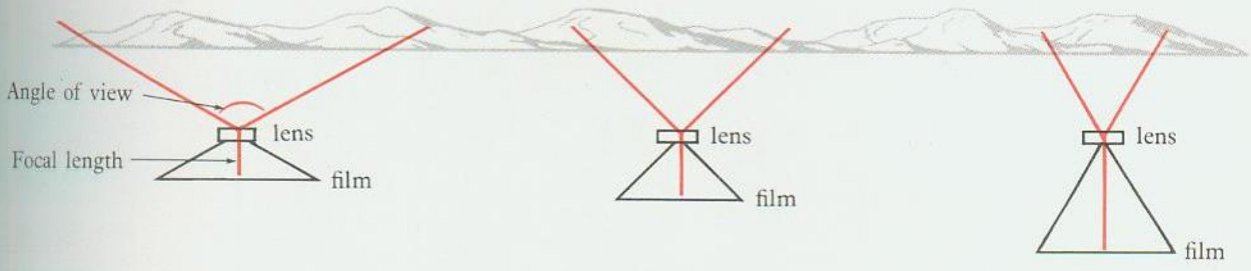
The "normal" lens takes a picture close to what the photographer sees. The "wide angle" lens takes a picture with a broader view of the scene. The telephoto lens actually magnifies the objects.

Photographers must understand the "geometry of photography" to know what each lens does best. We will investigate two ideas: *the focal length* and *the angle of view.*

The focal length is the distance from the lens to the film in the camera when the lens is focused on a distant object. The angle of view is the angle formed by imaginary lines from the edges of the scene to the lens of the camera.



The following diagram illustrates the relationship between focal length and angle of view. The lens of the camera is in the same position in each drawing.



Wide angle Normal Telephoto

The diagram shows that the shorter the focal length, the larger the angle of view. Wide angle lenses have smaller focal lengths and "see more" of a scene than normal lenses. Telephoto lenses have longer focal lengths and "see less" of the scene than normal lenses.

You can determine the approximate angle of view for a given focal length with the following construction.

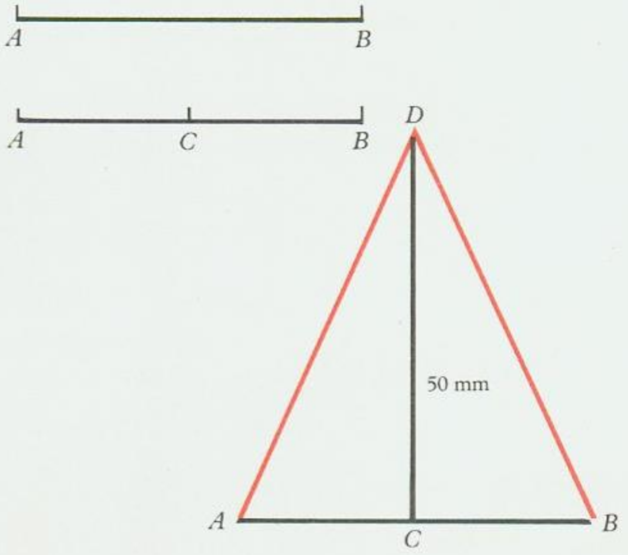
Step 1 Draw a segment $\overline{AB}$ with length of 43 mm. This is the length of the diagonal of a picture negative of ordinary film for a widely used camera.

Step 2 Bisect $\overline{AB}$. Call the midpoint C.

Step 3 Construct $\overline{DC} \perp \overline{AB}$, so that $\overline{DC}$ has length equal to the focal length of the lens. (For example, 50 mm)

Step 4 Draw $\overline{DA}$ and $\overline{DB}$.

Step 5 Measure $\angle ADB$ with a protractor to find the approximate angle of view. (For a focal length of 50 mm)



Complete the above construction for each of the following focal lengths.

1. 21 mm 2. 50 mm 3. 200 mm

For each focal length, measure to find the angle of view. A lens with focal length of 50 mm has an angle of view about the same as a human eye. This lens is a normal lens. Which focal length is for a wide angle lens? a telephoto lens?